A Note on Jackson and Yariv (2017)

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1 Introduction

In this short note, I present and prove two theorems that characterize environments in which there exists a representative agent. Both theorems are variants of the main results in the working paper "The Non-Existence of Representative Agents" by Jackson and Yariv (2017). In contrast to Jackson and Yariv (2017), I prove the two theorems without exploiting the properties of analytic functions. As a result, the theorems hold for more general real-valued utility functions.

2 Representative Agents with Private Allocations

There are $n \in \mathbb{N}$ agents, each of which is characterized by a parameter $\theta \in \Theta \subset \mathbb{R}^K$ for some natural number K. There is a private consumption good $x \in X \subset \mathbb{R}$. Each individual i's utility function $u_i : X \to \mathbb{R}$ is given by $u_i(x_i) \equiv u(x_i, \theta_i)\ \forall x_i \in X$ for some common function $u : X \times \Theta \to \mathbb{R}$. We say that there exists a *representative agent with private allocations* if $\forall(\lambda_1, ..., \lambda_n) \in \mathbb{R}^n_+$ with $\sum_{i=1}^n \lambda_i = 1$, and $\forall(\theta_1, ..., \theta_n) \in \Theta^n$, there exists $\bar{\theta} \in \Theta$ such that

$$\sum_{i=1}^{n} \lambda_i u(x_i, \theta_i) = u\left(\sum_{i=1}^{n} \lambda_i x_i, \bar{\theta}\right), \quad \forall(x_1, ..., x_n) \in X^n. \tag{1}$$

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Theorem 1 *Suppose that X is convex, Θ is connected, and u is differentiable in its first argument and is continuous in its second argument. Then, there exists a representative agent with private allocations if and only if there exist a constant $\alpha \in \mathbb{R}$ and a continuous function $v : \Theta \to \mathbb{R}$ such that $u(x, \theta) = \alpha x + v(\theta)$ for all $(x, \theta) \in X \times \Theta$.*

Proof. (*If statement*) Suppose that $u(x, \theta) = \alpha x + v(\theta)$, where v is continuous. Fix an arbitrary weight vector $(\lambda_1, ..., \lambda_n) \in \mathbb{R}^n_+$ with $\sum_{i=1}^n \lambda_i = 1$, and a preference profile $(\theta_1, ..., \theta_n) \in \Theta^N$. Since Θ is connected and v is continuous, the intermediate value theorem implies that there exists $\bar{\theta}$ such that

$$v\left(\bar{\theta}\right) = \sum_{i=1}^{n} \lambda_i v(\theta_i).$$

Hence, for all $(x_1, ..., x_n) \in X^n$, we have

$$\sum_{i=1}^{n} \lambda_i u(x_i, \theta_i) = \alpha \sum_{i=1}^{n} \lambda_i x_i + \sum_{i=1}^{n} \lambda_i v(\theta_i) = \alpha \sum_{i=1}^{n} \lambda_i x_i + v\left(\bar{\theta}\right) = u\left(\sum_{i=1}^{n} \lambda_i x_i, \bar{\theta}\right).$$

(*Only-if statement*) Suppose that a representative agent exists. Fix a weight vector $(\lambda_1, ..., \lambda_n) \in \mathbb{R}^n_+$. Without loss of generality, assume that $\lambda_i > 0$ for all $i = 1, ..., n$. The existence of a representative agent implies that for all $(\theta_1, ..., \theta_n) \in \Theta$, there exists $\bar{\theta} \in \Theta$ such that (1) holds for all $(x_1, ..., x_n) \in X^n$. Differentiating with respect to x_i for both sides of (1), we obtain that

$$\lambda_i u_1(x_i, \theta_i) = \lambda_i u_1\left(\sum_{i=1}^{n} \lambda_i x_i, \bar{\theta}\right), \tag{2}$$

where u_1 denotes the partial derivative of function u with respect to its first argument, and $u_1\left(\sum_{i=1}^n \lambda_i x_i, \bar{\theta}\right)$ is well-defined because the convexity of X implies that $\sum_{i=1}^n \lambda_i x_i \in X$. Since (2) holds for all i and $\lambda_i > 0$ $\forall i = 1, ..., n$, we can divide both sides of (2) by λ_i and further conclude that that for all $(\theta_1, ..., \theta_n) \in \Theta$, we must have

$$u_1(x_i, \theta_i) = u_1(x_j, \theta_j) \tag{3}$$

for all $x_i, x_j \in X$ and $i, j = 1, ..., n$. Therefore, there must exist $\alpha \in \mathbb{R}$ such that $u_1(x, \theta) = \alpha$

for all $(x,\theta) \in X \times \Theta$. Integrating u_1 with respect to x, we further obtain that $u(x,\theta) = \alpha x + v(\theta)$ for some $v : \Theta \to \mathbb{R}$. The continuity of v immediately follows from the continuity of u. $\square$

We next state the main result on the existence of a representative agent with private allocations in Jackson and Yariv (2017).

Theorem 1' (Jackson and Yariv, 2017) *Suppose that X is either open or compact, $\Theta \subset \mathbb{R}$ is compact, u is analytic, and there exists at least one $x^* \in X$ for which $u(x^*,\theta)$ is monotone in θ. Then, there exists a representative agent with private allocations if and only if there exist a constant $\alpha \in \mathbb{R}$ and an analytic and monotone function $v : \Theta \to \mathbb{R}$ such that $u(x,\theta) = \alpha x + v(\theta)$ for all $(x,\theta) \in X \times \Theta$*

The proof of Jackson and Yariv (2017)'s result relies on the properties of analytic functions, and it also makes use of the monotonicity of $u(x^*,\theta)$. In contrast, Theorem 1 of the current note makes a weaker assumption that u is differentiable in its first argument and is continuous in its second argument (it is weaker because any analytic function is infinitely differentiable). As its proof does not require the additional monotonicity assumption, Theorem 1 also holds beyond settings where Θ is one-dimensional.

3 Representative Agents with Common Allocations

We now turn to the case where there is only a public good, i.e. all the x_i are restricted to be equal to a common x. We say that there exists a representative agent with common allocations if $\forall(\lambda_1,...,\lambda_n) \in \mathbb{R}^n_+$ with $\sum_{i=1}^n \lambda_i = 1$, and $\forall(\theta_1,...,\theta_n) \in \Theta^n$, there exists $\bar{\theta} \in \Theta$ such that

$$\sum_{i=1}^{n} \lambda_i u(x,\theta_i) = u\left(x,\bar{\theta}\right), \quad \forall x \in X. \tag{4}$$

Theorem 2 *Suppose that $\Theta = [0,1]$, and there exists at least one $x^* \in X$ for which $u(x^*,\theta)$ is continuous and monotone in θ. Then, there exists a representative agent with common allocation if and only if there exist $v : \Theta \to [0,1]$, $f,g : X \to \mathbb{R}$ such that $u(x,\theta) = v(\theta)f(x) + g(x)$ for all $(x,\theta) \in X \times \Theta$, where v is continuous and monotone, and $f(x^*) \neq 0$.*

Proof. The proof of the if statement is analogous to that of Theorem 1, and we thus omit it to avoid repetition. For the only-if statement, suppose that a representative agent with common allocation exists. Without loss of generality, assume that $u(x^*, \theta)$ is increasing in θ, and hence $u(x^*, 1) > u(x^*, 0)$ and $u(x^*, 1) - u(x^*, 0) \geq u(x^*, \theta) - u(x^*, 0)$ for all $\theta \in \Theta$. Let

$$v(\theta) \equiv \frac{u(x^*, \theta) - u(x^*, 0)}{u(x^*, 1) - u(x^*, 0)}. \tag{5}$$

By construction, we have $v(\theta) \in [0, 1]$ and

$$u(x^*, \theta) = v(\theta)u(x^*, 1) + (1 - v(\theta))u(x^*, 0). \tag{6}$$

Since $u(x^*, \cdot)$ is monotone, there can be no $\theta' \in \Theta$ such that $\theta' \neq \theta$ and $u(x^*, \theta') = v(\theta)u(x^*, 1) + (1 - v(\theta))u(x^*, 0)$. Therefore, for the weight vector $(v(\theta), 1 - v(\theta), 0, ..., 0)$ and any preference profile $(\theta_1, ..., \theta_n)$ with $\theta_1 = 1$ and $\theta_2 = 0$, the utility function of the representative agent must be given by $u(\cdot, \theta)$. By definition, we have

$$u(x, \theta) = v(\theta)u(x, 1) + (1 - v(\theta))u(x, 0), \quad \forall x \in X. \tag{7}$$

Letting $f(x) = u(x, 1) - u(x, 0)$ and $g(x) = u(x, 0)$, we have

$$u(x, \theta) = v(\theta)f(x) + g(x), \ \forall x \in X, \tag{8}$$

which is the desired representation. The claims that v is continuous and monotone, and $f(x^*) \neq 0$ also immediately follow. □

We next state the main result on the existence of a representative agent with common allocations in Jackson and Yariv (2017).

Theorem 2' (Jackson and Yariv, 2017) *Suppose that X is either open or compact, $\Theta = [0, 1]$, u is analytic, and there exists at least one $x^* \in X$ for which $u(x^*, \theta)$ is monotone in θ. Then, there exists a representative agent with common allocations if and only if there exist analytic functions $v : \Theta \to \mathbb{R}$, $f, g : X \to \mathbb{R}$ such that $u(x, \theta) = v(\theta)f(x) + g(x)$ for all $(x, \theta) \in X \times \Theta$, where v is monotone and $f(x^*) \neq 0$.*

The proof of the above result from Jackson and Yariv (2017) again relies on the properties of analytic functions. In contrast, Theorem 2 of the current note, whose proof follows a similar argument as the one of Proposition 3 in Kushnir and Liu (2017), holds more generally for any real-valued utility function that is continuous and monotone in the preference parameter θ.

References

[1] Jackson, M. and Yariv, L. (2017): "The Non-Existence of Representative Agents," *Working paper.*

[2] Kushnir, A. and Liu, S. (2017): "On the Equivalence of Bayesian and Dominant Strategy Implementation for Environments with Non-Linear Utilities," *Working paper.*